A Hemimetric Extension of Simulation for Semi-Markov Decision Processes

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Mathias Ruggaard Pedersen, Giorgio Bacci, Kim Guldstrand Larsen, and Radu Mardare

{mrp, grbacci, kgl, mardare}@cs.aau.dk

Department of Computer Science, Aalborg University, Denmark

AALBORG UNIVERSITY
DENMARK

Agenda

Introduction

Introduction
Background

Semi-Markov decision processes have

- real-time behaviour,
- probabilistic behaviour, and
- non-deterministic behaviour.

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Semi-Markov decision processes have

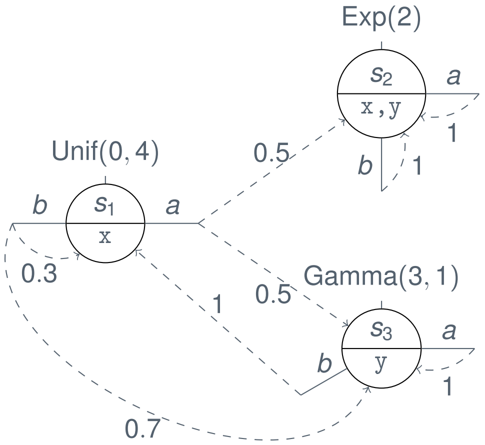
- real-time behaviour,
- probabilistic behaviour, and
- non-deterministic behaviour.

They have been used to model

- power plants,
- transportation infrastructure,
- revenue management systems,
- bridge maintenance,
- and more.

Introduction

Semi-Markov decision processes



Introduction
Semi-Markov decision processes

A Semi-Markov decision process is given by

- S a countable set of states,
- $\tau : S \times A \to \mathcal{D}(S)$ the transition function,
- $\rho : S \to \mathcal{D}(\mathbb{R}_{\geq 0})$ the residence-time function, and
- $L : S \to 2^{AP}$ the labelling function.

Introduction
Simulation

Definition

A relation $R \subseteq S \times S$ is a *simulation relation* if for all $(s_1, s_2) \in R$

- $L(s_1) = L(s_2)$,
- $F_{s_1}(t) \leq F_{s_2}(t)$ for all $t \in \mathbb{R}_{\geq 0}$, and
- for all $a \in A$ there exists a *coupling* $\Delta_a \colon S \times S \to [0, 1]$ between $\tau(s_1, a)$ and $\tau(s_2, a)$ such that
  - $\Delta_a(s, s') > 0$ implies $(s, s') \in R$,
  - $\tau(s_1, a)(s) = \sum_{s' \in S} \Delta_a(s, s')$ for all $s \in S$, and
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$s_1 \preceq s_2$ – s_2 simulates s_1

Introduction
Simulation (alternative)

Definition (alternative)

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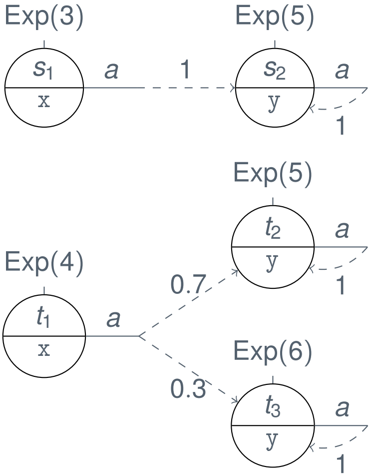
- for all $a \in A$ and $C \subseteq S$, $\tau(s_1, a)(C) \leq \tau(s_2, a)(R(C))$.

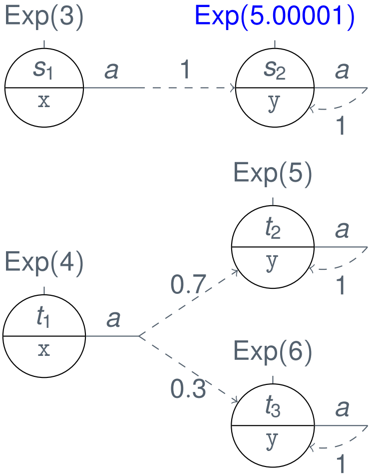
$R(C)$ upward closure of C.

Introduction
Example



$s_1 \preceq t_1$

Introduction
Example



$s_1 \not\preceq t_1$

Introduction

Quantitative behavioural relations

Jou and Smolka '90[1]: Pseudometrics rather than equivalences

[1] Chi-Chang Jou and Scott A. Smolka: Equivalences, congruences, and complete axiomatizations for probabilistic processes, CONCUR 1990

Introduction
Quantitative behavioural relations

Jou and Smolka '90[1]: Pseudometrics rather than equivalences

Now: Hemimetrics rather than preorders

[1] Chi-Chang Jou and Scott A. Smolka: Equivalences, congruences, and complete axiomatizations for probabilistic processes, CONCUR 1990

Hemimetric

Hemimetric
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We focus on the real-time behaviour of systems.

Hemimetric

A distance on CDFs

$F_2 \sqsubseteq F_1$ iff $F_1(t) \leq F_2(t)$ for all t – usual stochastic order

Hemimetric

A distance on CDFs

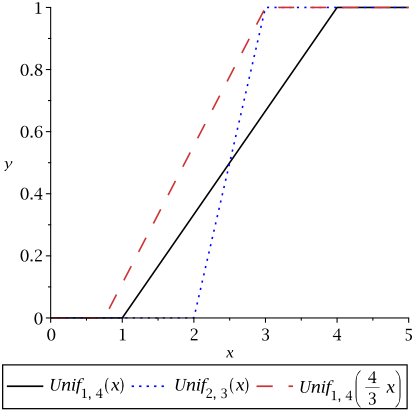
$F_2 \sqsubseteq F_1$ iff $F_1(t) \leq F_2(t)$ for all t – usual stochastic order

$F_2 \sqsubseteq_\varepsilon F_1$ iff $F_1(t) \leq F_2(\varepsilon \cdot t)$ for all t – ε-faster than

Hemimetric

Example

$$\mathrm{Unif}[1,4] \sqsubseteq_{\frac{4}{3}} \mathrm{Unif}[2,3]$$



Hemimetric

ε-simulation

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Hemimetric
ε-simulation

Definition

A relation $R \subseteq S \times S$ is a ε-simulation relation if for all $(s_1, s_2) \in R$

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$s_1 \preceq_\varepsilon s_2$ – s_2 ε-simulates s_1

Hemimetric

Distance

$$d(s_1, s_2) = \inf\{\varepsilon \geq 1 \mid s_1 \preceq_\varepsilon s_2\}$$

- $d(s_1, s_2) = 1$ iff $s_1 \preceq s_2$.
- $\log d(s_1, s_2)$ is a hemimetric.

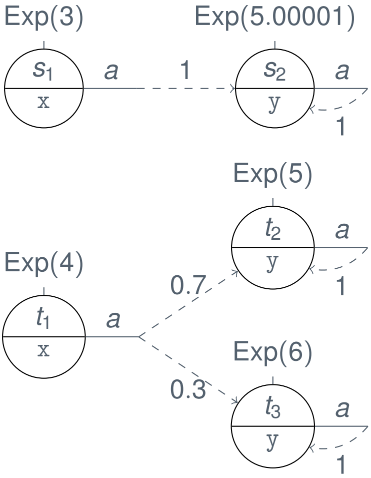
Hemimetric

Intuition

- ε is an *acceleration factor*.
- $s_1 \preceq_\varepsilon s_2$ means that s_2 simulates s_1 if we accelerate the real-time behaviour of s_2 by a factor ε.
- $d(s_1, s_2)$ is the infimum over such acceleration factors.

Hemimetric

Example



$s_1 \not\preceq t_1$

Hemimetric
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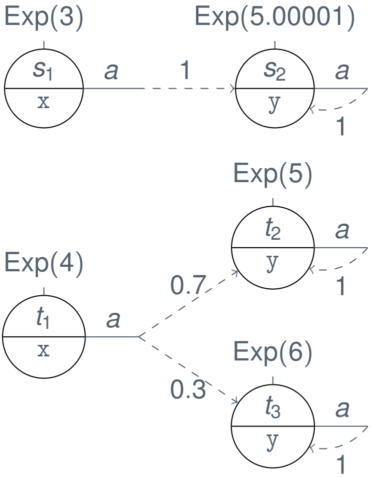

$$d(s_1, s_2) = \frac{5.00001}{5} = 1.000002$$

Computing the distance

Computing the distance
Definitions

$$c(F, G) = \inf\{\varepsilon \geq 1 \mid F \sqsubseteq_\varepsilon G\}$$

Note: We have closed-form solutions for $c(F, G)$ when F and G are Dirac, exponential, or uniform.

$$\mathcal{C}(M) = \{c(F_s, F_{s'}) \mid s, s' \in S\}$$

Computing the distance
The idea

Lemma

Let M be a finite SMDP. If $d(s_1, s_2) \neq \infty$, then

- *$s_1 \preceq_c s_2$ for some $c \in \mathcal{C}(M)$ and*
- *$d(s_1, s_2) = \min\{c \in \mathcal{C}(M) \mid s_1 \preceq_c s_2\}$.*

The lemma tells us:

- If there is no $c \in \mathcal{C}(M)$ such that $s_1 \preceq_c s_2$, then $d(s_1, s_2) = \infty$.
- If $s_1 \preceq_c s_2$ for some $c \in \mathcal{C}(M)$, then

$$d(s_1, s_2) = \min\{c \in \mathcal{C}(M) \mid s_1 \preceq_c s_2\}$$

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Let M be a finite SMDP. If $d(s_1, s_2) \neq \infty$, then

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$$d(s_1, s_2) = \min\{c \in \mathcal{C}(M) \mid s_1 \preceq_c s_2\}$$

We need to be able to decide whether $s_1 \preceq_c s_2$.

Computing the distance
Deciding ε-similarity

We adapt the algorithm from Baier et al.[2] for deciding similarity.

Theorem

For finite SMDPs, we can decide whether $s_1 \preceq_\varepsilon s_2$ in time $\mathcal{O}(n^2(f(I) + k) + (mn^7)/\log n)$, where

- $n = |S|$ *is the number of states,*
- $m = |A|$ *is the number of actions,*
- $k = |AP|$ *is the number of atomic propositions, and*
- $f(I)$ *is the complexity of computing* $c(F, G)$.

Note: $f(I)$ is constant when considering Dirac, exponential, and uniform distributions.

[2] Christel Baier, Bettina Engelen, and Mila E. Majster-Cederbaum: Deciding Bisimilarity and Similarity for Probabilistic Processes, J. Comput. Syst. Sci. 2000

Computing the distance

Algorithm

We can now use a bisection method to search through $c \in \mathcal{C}(M)$ and check whether $s_1 \preceq_c s_2$.

Order the elements of $\mathcal{C}(M) \setminus \{\infty\}$ such that $c_1 < c_2 < \cdots < c_n$;
if $s_1 \preceq_{c_1} s_2$ **then return** c_1 ;
else if $s_1 \not\preceq_{c_n} s_2$ **then return** ∞ ;
else
    $i \leftarrow 1, j \leftarrow n$;
    while $i < j$ **do**
        $h \leftarrow \left\lceil \frac{j-i}{2} \right\rceil$;
        if $s_1 \preceq_{c_{j-h}} s_2$ **then** $j \leftarrow j - h$;
        else $i \leftarrow i + h$;
    end
    return c_j;
end

Computing the distance
Complexity

Since the bisection method halves the number of remaining elements in each step, it iterates at most log n times.

Theorem

The simulation distance can be computed in time $\mathcal{O}(n^2(f(l) + k) + mn^7)$.

Compositionality

Compositionality
Composing SMDPs

Definition

$M_1 \parallel_\star M_2$ is given by

- $S = S_1 \times S_2$,
- $\tau((s_1, s_2), a)((s_1', s_2')) = \tau_1(s_1, a)(s_1') \cdot \tau_2(s_2, a)(s_2')$,
- $\rho((s_1, s_2)) = \star(\rho(s_1), \rho(s_2))$, and
- $L((s_1, s_2)) = L(s_1) \cup L(s_2)$.

$\star$ is a function for composing real-time behaviour.

Compositionality
Composing SMDPs

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$\star$ is a function for composing real-time behaviour.

- Maximum composition: $F_{\star(\mu,\nu)}(t) = \max(F_\mu(t), F_\nu(t))$.
- Product rate composition: $F_{\star(\mu,\nu)}(t) = \mathit{Exp}[\theta \cdot \theta'](t)$.
- Minimum rate composition: $F_{\star(\mu,\nu)}(t) = \mathit{Exp}[\min(\theta, \theta')](t)$.
- Maximum rate composition: $F_{\star(\mu,\nu)}(t) = \mathit{Exp}[\max(\theta, \theta')](t)$.

Compositionality

Non-expansiveness

Definition

$\star$ is *monotonic* if $F_\mu \sqsubseteq_\varepsilon F_\nu$ implies $F_{\star(\mu,\eta)} \sqsubseteq_\varepsilon F_{\star(\nu,\eta)}$.

Theorem

For finite SMDPs and monotonic $\star$,

$$d(s_1, s_2) \leq \varepsilon \quad \textit{implies} \quad d(s_1 \parallel_\star s_3, s_2 \parallel_\star s_3) \leq \varepsilon.$$

Logic

Logic
Timed Markovian Logic

$$\mathrm{TML}: \quad \varphi, \varphi' ::= \alpha \mid \neg\alpha \mid \ell_p t \mid m_p t \mid L_p^a \varphi \mid M_p^a \varphi \mid \varphi \wedge \varphi' \mid \varphi \vee \varphi'$$

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$s \models \alpha$	iff	$\alpha \in L(s)$	$s \models \ell_p t$	iff	$F_s(t) \geq p$
$s \models \neg\alpha$	iff	$\alpha \notin L(s)$	$s \models m_p t$	iff	$F_s(t) \leq p$
$s \models \varphi \wedge \varphi'$	iff	$s \models \varphi$ and $s \models \varphi'$	$s \models L_p^a \varphi$	iff	$\tau(s,a)([\![\varphi]\!]) \geq p$
$s \models \varphi \vee \varphi'$	iff	$s \models \varphi$ or $s \models \varphi'$	$s \models M_p^a \varphi$	iff	$\tau(s,a)([\![\varphi]\!]) \leq p$

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$s \models \neg\alpha$	iff	$\alpha \notin L(s)$	$s \models m_p t$	iff	$F_s(t) \leq p$
$s \models \varphi \wedge \varphi'$	iff	$s \models \varphi$ and $s \models \varphi'$	$s \models L_p^a \varphi$	iff	$\tau(s, a)(\llbracket \varphi \rrbracket) \geq p$
$s \models \varphi \vee \varphi'$	iff	$s \models \varphi$ or $s \models \varphi'$	$s \models M_p^a \varphi$	iff	$\tau(s, a)(\llbracket \varphi \rrbracket) \leq p$

$$\mathrm{TML}^{\geq}: \quad \varphi ::= \alpha \mid \neg\alpha \mid \ell_p t \mid L_p^a \varphi \mid \varphi \wedge \varphi' \mid \varphi \vee \varphi'$$

$$\mathrm{TML}^{\leq}: \quad \varphi ::= \alpha \mid \neg\alpha \mid m_p t \mid M_p^a \varphi \mid \varphi \wedge \varphi' \mid \varphi \vee \varphi'$$

Logic
ε-perturbation

$(\varphi)_\varepsilon$ is defined inductively as

$$
\begin{aligned}
(\alpha)_\varepsilon &= \alpha & (\ell_p t)_\varepsilon &= \ell_p \varepsilon \cdot t \\
(\neg\alpha)_\varepsilon &= \neg\alpha & (m_p t)_\varepsilon &= m_p \varepsilon \cdot t \\
(\varphi \wedge \varphi')_\varepsilon &= (\varphi)_\varepsilon \wedge (\varphi')_\varepsilon & (L_p^a \varphi)_\varepsilon &= L_p^a (\varphi)_\varepsilon \\
(\varphi \vee \varphi')_\varepsilon &= (\varphi)_\varepsilon \vee (\varphi')_\varepsilon & (M_p^a \varphi)_\varepsilon &= M_p^a (\varphi)_\varepsilon
\end{aligned}
$$

Logic
Logical characterisation

Theorem

For finite SMDPs the following holds.

- $d(s_1, s_2) \leq \varepsilon$ *if and only if* $\forall \varphi \in \mathrm{TML}^{\geq}. s_1 \models \varphi \implies s_2 \models (\varphi)_{\varepsilon}$.
- $d(s_1, s_2) \leq \varepsilon$ *if and only if* $\forall \varphi \in \mathrm{TML}^{\leq}. s_2 \models \varphi \implies s_1 \models (\varphi)_{\varepsilon}$.

Logic

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For finite SMDPs the following holds.

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- $d(s_1, s_2) \leq \varepsilon$ *if and only if* $\forall \varphi \in \mathrm{TML}^{\leq}. s_2 \models \varphi \implies s_1 \models (\varphi)_\varepsilon$.

Corollary

For finite SMDPs we have

$$s_1 \sim s_2 \text{ if and only if } \forall \varphi \in \mathrm{TML}. s_1 \models \varphi \iff s_2 \models \varphi.$$

Conclusion

Conclusion
Summary

- We have introduced a quantitative ε-simulation relation for SMDPs based on the usual stochastic order.
- ε-simulation induces a (multiplicative) hemimetric.
- The hemimetric is computable – polynomial time for Dirac, exponential, and uniform distributions.
- Parallel composition is non-expansive w.r.t. the hemimetric.
- The distance is characterised by a timed extension of Markovian logic.

Conclusion
Open Problems

- What about infinite systems?
- Computing $c(F, G) = \inf\{\varepsilon \geq 1 \mid F \sqsubseteq_\varepsilon G\}$ for composition.
  - E.g. uniform distributions are not closed under composition.
- Topological properties of the hemimetric?

Conclusion

Question

- *Simulation:* for all $a \in A$ there exists a coupling $\Delta_a \colon S \times S \to [0,1]$ between $\tau(s_1, a)$ and $\tau(s_2, a)$ such that
  - $\Delta_a(s, s') > 0$ implies $(s, s') \in R$,
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- *Simulation (alternative):* for all $a \in A$ and $C \subseteq S$, $\tau(s_1, a)(C) \leq \tau(s_2, a)(R(C))$.

The two definitions are equivalent for *finite* SMDPs[3].

[3] Lijun Zhang: Decision algorithms for probabilistic simulations, PhD thesis, Saarland University 2009

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The two definitions are equivalent for *finite* SMDPs[3].

Question: What about *infinite* SMDPs?

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