A Detailed proofs

Proof (Proof of Lemma 5). Clearly if $T_1 \subseteq T_2$, then if there exists $t \in T_1$ such that $s \xrightarrow{r} t$, then there also exists $t \in T_2$ such that $s \xrightarrow{r} t$. Hence $\theta(s)(T_1) \subseteq \theta(s)(T_2)$. □

Proof (Proof of Lemma 6).

- 1:

$$\begin{aligned}\theta(s)(T_1 \cup T_2) &= \{r \in \mathbb{R}_{\geq 0} \mid \exists t \in T_1 \cup T_2 \text{ such that } s \xrightarrow{r} t\} \\ &= \{r \in \mathbb{R}_{\geq 0} \mid \exists t \in T_1 \text{ such that } s \xrightarrow{r} t \\ &\quad \text{or } \exists t \in T_2 \text{ such that } s \xrightarrow{r} t\} \\ &= \{r \in \mathbb{R}_{\geq 0} \mid \exists t \in T_1 \text{ such that } s \xrightarrow{r} t\} \\ &\quad \cup \{r \in \mathbb{R}_{\geq 0} \mid \exists t \in T_2 \text{ such that } s \xrightarrow{r} t\} \\ &= \theta(s)(T_1) \cup \theta(s)(T_2) \quad .\end{aligned}$$

- 2: Similar to case 1.

□

Lemma 30. *Let $\mathcal{M} = (S, \rightarrow, \ell)$ be a WTS and let $s, t \in S$. $s \sim_W t$ if and only if $\theta(s)(T) = \theta(t)(T)$ for any $\sim_W$-equivalence class $T \subseteq S$.*

Proof. ($\Longrightarrow$) Assume $s \sim_W t$ and let $T \subseteq S$ be a $\sim_W$-equivalence class. If $r \in \theta(s)(T)$, then there exists some $s' \in T$ such that $s \xrightarrow{r} s'$. Because $s \sim_W t$, there must exist some $t' \in T$ such that $t \xrightarrow{r} t'$ and $s' \sim_W t'$. Since T_W is a $\sim$-equivalence class, this means that $t' \in T$, and hence $r \in \theta(t)(T)$. A similar argument shows that if $r \in \theta(t)(T)$, then $r \in \theta(s)(T)$.

($\Longleftarrow$) Assume $\theta(s)(T) = \theta(t)(T)$ for any $\sim_W$-equivalence class $T \subseteq S$. If $s \xrightarrow{r} s'$, then $r \in \theta(s)([s']_{\sim_W})$, and therefore $r \in \theta(t)([s']_{\sim_W})$, so $t \xrightarrow{r} t'$ for some $s' \sim_w t'$. A similar argument shows that if $t \xrightarrow{r} t'$, then $s \xrightarrow{r} s'$ for some $s' \sim_W t'$. Hence $s \sim_W t$. □

Proof (Proof of Theorem 10). We show that any weighted bisimulation is also a bisimulation. Let $\mathcal{M} = (S, \rightarrow, \ell)$ be a WTS, and let $\mathcal{R} \subseteq S \times S$ be a weighted bisimulation relation. Let $s, t \in S$. We have that $\ell(s) = \ell(t)$, and by Lem. 30, we have that $\theta(s)(T) = \theta(t)(T)$ for any $\mathcal{R}$-equivalence class $T \subseteq S$. This implies that in particular $\theta^-(s)(T) = \theta^-(t)(T)$ and $\theta^+(s)(T) = \theta^+(t)(T)$. Hence $\mathcal{R}$ is a bisimulation relation.

By Ex. 9, the inclusion is strict. □

Lemma 31. *Given any WTS $\mathcal{M} = (S, \rightarrow, \ell)$, it holds that if $T_0 \supseteq T_1 \supseteq \ldots$ is a countable, decreasing sequence of subsets of S, then*

$$\theta(s)\left(\bigcap_i T_i\right) = \bigcap_i \theta(s)(T_i) \quad .$$

Proof. We first show that $\theta(s)(\bigcap_i T_i) = \emptyset$ iff $\bigcap_i \theta(s)(T_i) = \emptyset$. To this end, assume $\theta(s)(\bigcap_i T_i) \neq \emptyset$. Then there exists some $r \in \theta(s)(\bigcap_i T_i)$ which means that there exists $t \in \bigcap_i T_i$ such that $s \xrightarrow{r} t$. Hence, for all i we have $t \in T_i$ and $s \xrightarrow{r} t$. This means that $r \in \theta(s)(T_i)$ for all i, and thus $r \in \bigcap_i \theta(s)(T_i)$. Now assume $\bigcap_i \theta(s)(T_i) \neq \emptyset$. Then there must exist some $r' \in \bigcap_i \theta(s)(T_i)$, which implies that for all T_i there exists $t \in T_i$ such that $s \xrightarrow{r'} t$. This implies that there exists $t \in \bigcap_i T_i$ such that $s \xrightarrow{r'} t$, and hence $r' \in \theta(s)(\bigcap_i T_i)$, so $\theta(s)(\bigcap_i T_i) \neq \emptyset$.

Now assume that $\theta(s)(\bigcap_i T_i) \neq \emptyset$ and $\bigcap_i \theta(s)(T_i) \neq \emptyset$. Let $r' \in \theta(s)(\bigcap_i T_i)$. Then there exists $t \in S$ such that $t \in T_i$ for all T_i and $s \xrightarrow{r'} t$. This means that $r' \in \theta(s)(T_i)$ for all T_i, and hence $r' \in \bigcap_i \theta(s)(T_i)$.

Next assume towards a contradiction that $\theta(s)(\bigcap_i T_i) \subsetneq \bigcap_i \theta(s)(T_i)$, meaning that there exists some $r' \in \mathbb{R}_{\geq 0}$ such that $r' \in \bigcap_i \theta(s)(T_i)$ but $r' \notin \theta(s)(\bigcap_i T_i)$. $r' \in \bigcap_i \theta(s)(T_i)$ implies that that for all T_i there exists $t \in T_i$ such that $s \xrightarrow{r'} t$, which implies that there exists $t \in \bigcap_i T_i$ such that $s \xrightarrow{r'} t$. However, $r' \notin \theta(s)(\bigcap_i T_i)$ implies that for all $t \in \bigcap_i T_i$ we have $s \not\xrightarrow{r'} t$, which is a contradiction. □

Proof (Proof of Theorem 14). We first show that $s \sim t$ implies $\mathcal{M}, s \models \varphi$ if and only if $\mathcal{M}, t \models \varphi$ for all $\varphi \in \mathcal{L}$ by induction on φ. The boolean cases are trivial. If $\varphi = L_r\psi$, then we have $\theta^-(s)(\llbracket\psi\rrbracket) \geq r$, which implies that $\theta^-(s)(\llbracket\psi\rrbracket) \neq -\infty$. Assume towards a contradiction that $\theta^-(t)(\llbracket\psi\rrbracket) < r$. It can not be the case that $\theta^-(t)(\llbracket\psi\rrbracket) = -\infty$, hence it follows that $\llbracket\psi\rrbracket$ and $\theta(t)(\llbracket\psi\rrbracket)$ are non-empty, so there must exist some element $t' \in \llbracket\psi\rrbracket$ such that $\theta^-(t)(\llbracket\psi\rrbracket) \leq \theta^-(t)(\{t'\}) < r$. Since $\mathcal{R}$ is an equivalence relation, there must exists some $\mathcal{R}$-equivalence class T such that $t' \in T$. This means that $\{t'\} \subseteq T$, so that also $\theta^-(t)(T) \leq \theta^-(t)(\{t'\}) < r$. By the induction hypothesis we have that $T \subseteq \llbracket\psi\rrbracket$. Because $s \sim t$, we have that $\theta^-(s)(T) = \theta^-(t)(T) < r$, so by monotonicity we get $\theta^-(s)(\llbracket\psi\rrbracket) \leq \theta^-(s)(T) < r$, which is a contradiction. The M_r case is handled similarly.

For the reverse direction of the biconditional we have to show that if for all $\varphi \in \mathcal{L}$, $\mathcal{M}, s \models \varphi$ if and only if $\mathcal{M}, t \models \varphi$ then $s \sim t$. To this end, we define a relation $\mathcal{R}$ on S as

$$\mathcal{R} = \{(s,t) \in S \times S \mid \forall \varphi \in \mathcal{L}.\ \mathcal{M}, s \models \varphi \text{ iff } \mathcal{M}, t \models \varphi\} \ .$$

$\mathcal{R}$ is clearly an equivalence relation and $s\mathcal{R}t$.

It is clear that $\ell(s) = \ell(t)$. Next we show that $\theta^-(s)(T) = \theta^-(t)(T)$ and $\theta^+(s)(T) = \theta^+(t)(T)$ for any $\mathcal{R}$-equivalence class T. Let $T \subseteq S$ be an $\mathcal{R}$-equivalence class and let $\llbracket T\rrbracket$ denote the set of formulae satisfied by all the states in T, i.e.

$$\llbracket T\rrbracket = \{\varphi \in \mathcal{L} \mid \forall t' \in T.\ \mathcal{M}, t' \models \varphi\} \ .$$

Since $\mathcal{L}$ is countable, we can enumerate the formulae of $\llbracket T\rrbracket$ as $\llbracket T\rrbracket = \{\varphi_0, \varphi_1, \ldots\}$. For $i \in \mathbb{N}$ we define $\psi_0 = \varphi_0$ and $\psi_i = \psi_{i-1} \wedge \varphi_i$. We then have a decreasing sequence $\llbracket\psi_0\rrbracket \supseteq \llbracket\psi_1\rrbracket \supseteq \ldots$ such that $T = \bigcap_{i \in \mathbb{N}} \llbracket\psi_i\rrbracket$.

We will first show that $\theta(s)(T) = \emptyset$ if and only if $\theta(t)(T) = \emptyset$. This follows from the fact that all the image sets are assumed to be compact, and hence we can use Lem. 31 and Cantor's intersection theorem to deduce that $\theta(s)(\llbracket\psi_i\rrbracket) = \emptyset$ for some ψ_i, and we then use the fact that $\theta(s)(\llbracket\psi_i\rrbracket) = \emptyset$ iff $\mathcal{M}, s \models \neg L_0\psi_i$.

Now assume that $\theta(s)(T) \neq \emptyset$ and $\theta(t)(T) \neq \emptyset$. We need to show that $\theta^-(s)(T) = \theta^-(t)(T)$ and $\theta^+(s)(T) = \theta^+(t)(T)$. We do this by contradiction, which gives us four cases to consider: $\theta^-(s)(T) < \theta^-(t)(T)$, $\theta^-(s)(T) > \theta^-(t)(T)$, $\theta^+(s)(T) < \theta^+(t)(T)$, and $\theta^+(s)(T) > \theta^+(t)(T)$.

For the case of $\theta^-(s)(T) < \theta^-(t)(T)$, there exists $q \in \mathbb{Q}_{\geq 0}$ such that

$$\theta^-(s)(T) < q < \theta^-(t)(T) \quad ,$$

which implies that there exist j such that

$$\theta^-(s)(T) < q < \theta^-(t)(\llbracket\psi_j\rrbracket) \leq \theta^-(t)(T) \quad ,$$

and since $\theta^-(s)(\llbracket\psi_i\rrbracket) \leq \theta^-(s)(T)$ for any i by monotonicity, we get that $\mathcal{M}, s \not\models L_q\psi_j$ and $\mathcal{M}, s \models L_q\psi_j$, which is a contradiction. The other cases are handled similarly. □

Proof (Proof of Theorem 15). The soundness of each axiom is easy to show, and many of them use the distributive property from Lem. 6. Here we prove the soundness for a few of the more interesting axioms.

A3

Suppose $\mathcal{M}, s \models L_r\varphi \wedge L_q\psi$ implying that $\mathcal{M}, s \models L_r\varphi$ and $\mathcal{M}, s \models L_q\psi$, implying further that $\theta^-(s)(\llbracket\varphi\rrbracket) \geq r$ and $\theta^-(s)(\llbracket\psi\rrbracket) \geq q$.
By Lem. 6 we must have that

$$\theta(s)(\llbracket\varphi \vee \psi\rrbracket) = \theta(s)(\llbracket\varphi\rrbracket \cup \llbracket\psi\rrbracket) = \theta(s)(\llbracket\varphi\rrbracket) \cup \theta(s)(\llbracket\psi\rrbracket)$$

and because $\theta^-(s)(\llbracket\varphi\rrbracket) \geq r$ and $\theta^-(s)(\llbracket\psi\rrbracket) \geq q$ we must have

$$\theta^-(s)(\llbracket\varphi \vee \psi\rrbracket) = \inf \theta(s)(\llbracket\varphi\rrbracket) \cup \theta(s)(\llbracket\psi\rrbracket) \geq \min\{r, q\}$$

implying $\mathcal{M}, s \models L_{\min\{r,q\}}\varphi \vee \psi$.

A4

Suppose $\mathcal{M}, s \models L_r(\varphi \vee \psi)$ implying that

$$\theta^-(s)(\llbracket\varphi \vee \psi\rrbracket) = \inf \theta(s)(\llbracket\varphi\rrbracket) \cup \theta(s)(\llbracket\psi\rrbracket) \geq r \quad .$$

This implies that at least one of $\theta(s)(\llbracket\varphi\rrbracket)$ and $\theta(s)(\llbracket\psi\rrbracket)$ is non-empty. If $\theta(s)(\llbracket\varphi\rrbracket) \neq \emptyset$, then $\theta^-(s)(\llbracket\varphi\rrbracket) \geq r$, and also if $\theta(s)(\llbracket\psi\rrbracket) \neq \emptyset$, then $\theta^-(s)(\llbracket\psi\rrbracket) \geq r$, so at least one of $\mathcal{M}, s \models L_r\varphi$ and $\mathcal{M}, s \models L_r\psi$ must hold. Hence $\mathcal{M}, s \models L_r\varphi \vee L_r\psi$.

A6

Suppose $\mathcal{M}, s \models L_{r+q}\varphi$ implying that

$$\theta^-(s)(\llbracket\varphi\rrbracket) = \inf \theta(s)(\llbracket\varphi\rrbracket) \geq r + q \quad .$$

It is clear that $\inf \theta(s)(\llbracket\varphi\rrbracket) \leq \sup \theta(s)(\llbracket\varphi\rrbracket)$, so

$$\theta^{+}(s)(\llbracket\varphi\rrbracket) = \sup \theta(s)(\llbracket\varphi\rrbracket) \geq \inf \theta(s)(\llbracket\varphi\rrbracket) \geq r + q > r \ .$$

Therefore, it cannot be the case that $\mathcal{M}, s \models M_r\varphi$ and thus $\mathcal{M}, s \models \neg M_r\varphi$.

R1

Suppose $\models \varphi \to \psi$ implying that $\llbracket\varphi\rrbracket \subseteq \llbracket\psi\rrbracket$, implying further, by the monotonicity of θ, that $\theta(s)(\llbracket\varphi\rrbracket) \subseteq \theta(s)(\llbracket\psi\rrbracket)$. Suppose further that $\mathcal{M}, s \models L_r\psi \wedge L_0\varphi$ implying $\mathcal{M}, s \models L_r\psi$ and $\mathcal{M}, s \models L_0\varphi$, implying further that

$$\theta^{-}(s)(\llbracket\psi\rrbracket) = \inf \theta(s)(\llbracket\psi\rrbracket) \geq r \quad \text{and} \quad \theta(s)(\llbracket\varphi\rrbracket) \neq \emptyset \ .$$

Since $\theta(s)(\llbracket\varphi\rrbracket)$ is non-empty, we then get that

$$\inf \theta(s)(\llbracket\varphi\rrbracket) \geq \inf \theta(s)(\llbracket\psi\rrbracket) \geq r \ ,$$

which means that $\mathcal{M}, s \models L_r\varphi$.

□

Lemma 32. *From the axioms listed in Tab. 1 we can derive the following theorems:*

(T1): $\vdash (L_r\varphi \wedge L_q\psi \wedge L_0(\varphi \wedge \psi)) \to L_{\max\{r,q\}}(\varphi \wedge \psi)$
(T1′): $\vdash (M_r\varphi \wedge M_q\psi \wedge L_0(\varphi \wedge \psi)) \to M_{\min\{r,q\}}(\varphi \wedge \psi)$
(T2): $\vdash \varphi \leftrightarrow \psi \implies \vdash L_r\varphi \leftrightarrow L_r\psi$
(T2′): $\vdash \varphi \leftrightarrow \psi \implies \vdash M_r\varphi \leftrightarrow M_r\psi$
(T3): $\vdash \neg L_r\bot, \quad r > 0$
(T4): $\vdash \varphi \to \bot \implies \vdash \neg L_r\varphi, \quad r \geq 0$
(T5): $\vdash M_r(\varphi \vee \psi) \to M_r\varphi \vee M_r\psi$

Proof.

T1 Axiom R1 implies

$$\vdash \neg L_q(\varphi \wedge \psi) \to (\neg L_q\varphi \vee \neg L_0(\varphi \wedge \psi)) \ ,$$

so also

$$\vdash \neg L_q(\varphi \wedge \psi) \to (\neg L_q\varphi \vee \neg L_0(\varphi \wedge \psi) \vee \neg L_r\psi) \ .$$

This is equivalent to

$$\vdash (L_r\varphi \wedge L_q\psi \wedge L_0(\varphi \wedge \psi)) \to L_q(\varphi \wedge \psi) \ .$$

T1′ Similar to T1.

T2 Suppose $\vdash \varphi \leftrightarrow \psi$. If $\vdash L_r\varphi$, then axiom A2 gives $\vdash L_0\varphi$, and axiom R2 then gives $\vdash L_0\psi$. Finally, axiom R1 then implies $\vdash L_r\psi$. A similar argument shows that if $\vdash L_r\psi$, then $\vdash L_r\varphi$. Hence $\vdash L_r\varphi \leftrightarrow L_r\psi$.

T2′ Similar to T2.

T3 From axiom A1 we know that $\vdash \neg L_0\bot$ which, by the contrapositive of A2, implies $\neg L_r\bot$ for any $r > 0$.

T4 Suppose $\vdash \varphi \to \bot$. We know for any $\psi \in \mathcal{L}$ that $\vdash \bot \to \psi$ and therefore $\vdash \varphi \to \bot \implies \vdash \varphi \leftrightarrow \bot$. From A1 we know that $\vdash \neg L_0 \bot$ and from T3 that $\vdash \neg L_r \bot$ for any $r > 0$ implying, by T2, that $\vdash L_r \varphi$ for any $r \geq 0$.

T5 Assume $\vdash M_r(\varphi \vee \psi)$. By axiom A7 we get $\vdash L_0(\varphi \vee \psi)$, and axiom A4 then gives $\vdash L_0 \varphi \vee L_0 \psi$. Since $\vdash \varphi \to \varphi \vee \psi$ and $\vdash \psi \to \varphi \vee \psi$, axiom R1′ then gives $\vdash M_r \varphi \vee M_r \psi$.

□

Proof (Proof of Lemma 18). Assume towards a contradiction that $L(u, v) = \emptyset$ and $M(u, v) \neq \emptyset$. Then we have $\neg L_0 (\!|v|\!) \in u$ and there exists some $r \in Q_\rho$ such that $M_r (\!|v|\!) \in u$. However, by axiom A7, this implies that $L_0 (\!|v|\!) \in u$, which is a contradiction. □

Lemma 33. *For any ultrafilters $u, v \in \mathcal{U}[\rho]$, if $L(u, v) \neq \emptyset$ and $M(u, v) \neq \emptyset$, then* $\max L(u, v) \leq \min M(u, v)$.

Proof. Assume towards a contradiction that $\max L(u, v) > \min M(u, v)$. Then there exist $q, q' \in Q_\rho$ such that $q > q'$, $L_q (\!|v|\!) \in u$ and $M_{q'} (\!|v|\!) \in u$. Since $q > q'$, axiom A6 gives $\neg M_{q'} (\!|v|\!) \in u$, which is a contradiction. □

Lemma 34. *For any consistent formula $\varphi \in \mathcal{L}[\rho]$, if $[\mathcal{M}_\rho, u \models \varphi$ iff $\varphi \in u]$, then*

$$\bigvee_{v \in [\![\varphi]\!]} (\!|v|\!) \in u \quad \textit{iff} \quad \varphi \in u \ .$$

Proof. Suppose $\bigvee_{v \in [\![\varphi]\!]} (\!|v|\!) \in u$. Assume towards a contradiction that $\neg (\!|v|\!) \in u$ for all $v \in [\![\varphi]\!]$. Then, since u is an ultrafilter, we must have $\bigwedge_{v \in [\![\varphi]\!]} \neg (\!|v|\!) \in u$, which means that $\neg \bigvee_{v \in [\![\varphi]\!]} (\!|v|\!) \in u$, which is a contradiction. Hence there exists some $v' \in [\![\varphi]\!]$ such that $(\!|v'|\!) \in u$. If $\psi \in v'$, then $\vdash (\!|v'|\!) \to \psi$, so $\psi \in u$ because u is an ultrafilter. Since $v' \in [\![\varphi]\!]$, we have by assumption that $\varphi \in v'$, so we get $\varphi \in u$.

Suppose $\varphi \in u$, which by assumption means that $u \in [\![\varphi]\!]$, so $\vdash (\!|u|\!) \to \bigvee_{v \in [\![\varphi]\!]} (\!|v|\!)$. Since u is an ultrafilter, we have $(\!|u|\!) \in u$, and hence $\bigvee_{v \in [\![\varphi]\!]} (\!|v|\!) \in u$. □

Proof (Proof of Lemma 19). The proof is by induction on the structure of φ. The boolean cases are trivial. For the case $\varphi = L_r \psi$, we proceed as follows.

($\implies$) Assume $\mathcal{M}_\rho, u \models L_r \psi$, meaning that $\theta^-(u)([\![\psi]\!]) \geq r$. It can not be the case that $\theta(u)([\![\psi]\!]) = \emptyset$, because otherwise $\theta^-(u)([\![\psi]\!]) = -\infty$, and we have assumed $\theta^-(u)([\![\psi]\!]) \geq r$. It also can not be the case that $[\![\psi]\!] = \emptyset$, because otherwise $\theta(u)([\![\psi]\!]) = \emptyset$. We can partition all the ultrafilters $v \in [\![\psi]\!]$ as follows. Let $E = \{v \in [\![\psi]\!] \mid L(u, v) = \emptyset\}$ and $N = \{v \in [\![\psi]\!] \mid L(u, v) \neq \emptyset\}$. We then get that $E \cap N = \emptyset$, $E \cup N = [\![\psi]\!]$, $\neg L_0 (\!|v|\!) \in u$ for all $v \in E$, and $L_r (\!|v|\!) \in u$ for all $v \in N$. Because u is an ultrafilter, we then have

$$\bigwedge_{v \in E} \neg L_0 (\!|v|\!) \wedge \bigwedge_{v \in N} L_r (\!|v|\!) \in u \ .$$

By axiom A3, this implies

$$\bigwedge_{v \in E} \neg L_0 (\!| v |\!) \wedge L_r \bigvee_{v \in N} (\!| v |\!) \in u \ \ .$$

Then axiom A5 gives

$$L_r \bigvee_{v \in [\![\psi]\!]} (\!| v |\!) \in u \ \ .$$

By the induction hypothesis, T2, and Lem. 34, we then get $L_r \psi \in u$.

($\Longleftarrow$) Let $L_r \psi \in u$. It follows from A1, A2, and R2 that ψ is consistent. Hence, by the induction hypothesis, $[\![\psi]\!]$ is non-empty. We first show that $\theta(u)([\![\psi]\!]) \neq \emptyset$. Assume therefore towards a contradiction that $\theta(u)([\![\psi]\!]) = \emptyset$. Then for all $v \in [\![\psi]\!]$, we must have that case 3 holds, and hence $L(u,v) = \emptyset$, meaning $\neg L_r (\!| v |\!) \in u$ for all $v \in [\![\psi]\!]$. Since there are finitely many $v \in [\![\psi]\!]$, we can enumerate them as $v_1, v_2, \ldots, v_n$. Then, since u is an ultrafilter, we have

$$\neg L_r (\!| v_1 |\!) \wedge \neg L_r (\!| v_2 |\!) \wedge \cdots \wedge \neg L_r (\!| v_n |\!) \in u \ \ .$$

By De Morgan's law, this is equivalent to

$$\neg (L_r (\!| v_1 |\!) \vee L_r (\!| v_2 |\!) \vee \cdots \vee L_r (\!| v_n |\!)) \in u \ \ .$$

The contrapositive of axiom A4 then gives that

$$\neg L_r ((\!| v_1 |\!) \vee (\!| v_2 |\!) \vee \cdots \vee (\!| v_n |\!)) \in u \ \ ,$$

and by the induction hypothesis, T2, and Lem. 34, this is equivalent to $\neg L_r \psi \in u$, which is a contradiction.

Now assume towards a contradiction that $\theta^-(u)([\![\psi]\!]) < r$. Then there exists some $v \in [\![\psi]\!]$ such that $\theta^-(u)(\{v\}) < r$ and case 1 or case 2 holds. In either case we have $\max L(u,v) < r$ and hence there exists some $q \in Q_\rho$ such that $L_q (\!| v |\!) \in u$, which implies $L_0 (\!| v |\!) \in u$ by axiom A2. By the induction hypothesis, $\psi \in v$, which means that $\vdash (\!| v |\!) \rightarrow \psi$. Axiom R1 then gives $L_r (\!| v |\!) \in u$, but this is a contradiction since $\max L(u,v) < r$.

The M_r case is similar, using axiom A7 instead of A2 to derive $L_0 \psi \in u$. $\square$

Proof (Proof of Theorem 20). Since $\varphi \in \mathcal{L}$ is consistent, the Rasiowa-Sikorski lemma [1] guarantees that there exists an ultrafilter $u \in \mathcal{U}[\varphi]$ such that $\varphi \in u$. By the truth lemma, this means that $\mathcal{M}_\varphi, u \models \varphi$, and by construction, $\mathcal{M}_\varphi$ is a finite model. $\square$

Proof (Proof of Theorem 21).

$$\models \varphi \quad \text{implies} \quad \vdash \varphi$$

is equivalent to

$$\nvdash \varphi \quad \text{implies} \quad \not\models \varphi \ \ ,$$

which is equivalent to

the consistency of $\neg\varphi$ implies the existence of a model for $\neg\varphi$,

and this is guaranteed by the truth lemma. $\square$

Lemma 35. *For any maximal set $\Gamma \subseteq \mathcal{L}[\rho]$, we have that*

1. $\varphi \in \Gamma$ *and* $\psi \in \Gamma$ *implies* $\varphi \wedge \psi \in \Gamma$, *and*
2. $\varphi \in \Gamma$ *or* $\psi \in \Gamma$ *implies* $\varphi \vee \psi \in \Gamma$.

Proof. To prove the first part, assume $\varphi \in \Gamma$ and $\psi \in \Gamma$. By P1, we know that either $\varphi \wedge \psi \in \Gamma$ or $\neg(\varphi \wedge \psi) \in \Gamma$. If $\neg(\varphi \wedge \psi) \in \Gamma$, then by P3 we must have $\neg\varphi \in \Gamma$ or $\neg\psi \in \Gamma$, which is a contradiction.

To prove the second part, assume $\varphi \in \Gamma$ or $\psi \in \Gamma$. By P1, we know that either $\varphi \vee \psi \in \Gamma$ or $\neg(\varphi \vee \psi) \in \Gamma$. If $\neg(\varphi \vee \psi) \in \Gamma$, then by P2 we must have $\neg\varphi \in \Gamma$ and $\neg\psi \in \Gamma$, which is a contradiction. □

Lemma 36. *For arbitrary maximal set of formulae $\Gamma \subseteq \mathcal{L}[\rho]$ it holds that*

$$\varphi \leftrightarrow \psi \in \Gamma \quad \textit{implies} \quad L_r\varphi \in \Gamma \textit{ iff } L_r\psi \in \Gamma \ .$$

Proof. Let $\Gamma \in \mathcal{L}[\rho]$ be a maximal set of formulae and suppose $\varphi \leftrightarrow \psi \in \Gamma$.

If $L_r\varphi \in \Gamma$ we have, by Q2, that $L_0\varphi \in \Gamma$ implying, by Q9, that $L_0\psi \in \Gamma$. We thus have $L_r\varphi \in \Gamma$ and $L_0\psi \in \Gamma$ implying, by P1 and P2, that $L_r\varphi \wedge L_0\psi \in \Gamma$ and therefore, by Q8, that $L_r\psi \in \Gamma$.

If $L_r\psi \in \Gamma$ we have, by Q2, that $L_0\psi \in \Gamma$ implying, by Q9, that $L_0\varphi \in \Gamma$. We thus have $L_r\psi \in \Gamma$ and $L_0\varphi \in \Gamma$ implying, by P1 and P2, that $L_r\psi \wedge L_0\varphi \in \Gamma$ and therefore, by Q8, that $L_r\varphi \in \Gamma$. □

For arbitrary $\varphi \in \mathcal{L}[\rho]$, Ω_φ denotes the collection of all maximal sets Γ in the language of ρ such that φ is contained in Γ, i.e.

$$\Omega_\varphi = \{\Gamma \subseteq \mathcal{L}[\rho] \mid \varphi \in \Gamma \text{ and } \Gamma \text{ is maximal}\} \ .$$

Lemma 37. *For arbitrary formula $\varphi \in \mathcal{L}[\rho]$ and maximal set of formulae $\Gamma \subseteq \mathcal{L}[\rho]$ it holds that*

$$\bigvee_{\Gamma_\varphi \in \Omega_\varphi} (\!|\Gamma_\varphi|\!) \in \Gamma \quad \textit{iff} \quad \varphi \in \Gamma \ .$$

Proof. Let $\varphi \in \mathcal{L}[\rho]$ be a formula and $\Gamma \subseteq \mathcal{L}[\rho]$ a maximal set of formulae.

Suppose $\bigvee_{\Gamma_\varphi \in \Omega_\varphi} (\!|\Gamma_\varphi|\!) \in \Gamma$ implying, by P3, the existence of $\Gamma_\varphi \in \Omega_\varphi$ such that $(\!|\Gamma_\varphi|\!) \in \Gamma$ implying further for all $\psi \in \Gamma_\varphi$ that $\psi \in \Gamma$. Because $\Gamma_\varphi \in \Omega_\varphi$ we must have $\varphi \in \Gamma_\varphi$ and therefore $\varphi \in \Gamma$.

Suppose $\varphi \in \Gamma$ implying that $\Gamma \in \Omega_\varphi$. By P1 we must have either $(\!|\Gamma|\!) \in \Gamma$ or $\neg(\!|\Gamma|\!) \in \Gamma$. $\neg(\!|\Gamma|\!) \in \Gamma$ is equivalent to $\bigvee_{\psi \in \Gamma} \neg\psi \in \Gamma$ implying, by P3, the existence of a formula $\psi \in \Gamma$ such that $\neg\psi \in \Gamma$ which, by P1, is a contradiction and therefore $(\!|\Gamma|\!) \in \Gamma$ implying, by P1 and P2, that $\bigvee_{\Gamma_\varphi \in \Omega_\varphi} (\!|\Gamma_\varphi|\!) \in \Gamma$. □

Lemma 38. *For arbitrary maximal sets of formulae $\Gamma, \Gamma' \in \mathcal{L}[\rho]$ it holds that*

$$M(\Gamma, \Gamma') \neq \emptyset \quad \textit{implies} \quad \max L(\Gamma, \Gamma') \leq \min M(\Gamma, \Gamma')$$

Proof. Let $\Gamma, \Gamma' \in \mathcal{L}[\rho]$ be maximal sets of formulae and suppose $M(\Gamma, \Gamma') \neq \emptyset$. There must exist a rational number $q \in \mathbb{Q}_{\geq 0}$ such that $q \in M(\Gamma, \Gamma')$ implying that $M_q (\!|\Gamma'|\!) \in \Gamma$ implying further, by Q7, that $L_0 (\!|\Gamma'|\!) \in \Gamma$ and therefore $L(\Gamma, \Gamma') \neq \emptyset$.

Suppose towards a contradiction that $\max L(\Gamma, \Gamma') > \min M(\Gamma, \Gamma')$ implying the existence of a rational number $q \in \mathbb{Q}_{\geq 0}$ such that $\max L(\Gamma, \Gamma') > q > \min M(\Gamma, \Gamma')$. $q > \min M(\Gamma, \Gamma')$ implies the existence of a rational number $r \in \mathbb{Q}_{\geq 0}$ such that $r < q$ and $r \in M(\Gamma, \Gamma')$ which further implies $M_r (\!|\Gamma'|\!) \in \Gamma$. $\max L(\Gamma, \Gamma') > q$ implies the existence of a rational number $r' \in \mathbb{Q}_{\geq 0}$ such that $r' > q$ and $r' \in L(\Gamma, \Gamma')$ implying further that $L_{r'} (\!|\Gamma'|\!) \in \Gamma$ which, by Q6, implies $\neg M_r (\!|\Gamma'|\!) \in \Gamma$. We thus have $M_r (\!|\Gamma'|\!) \in \Gamma$ and $\neg M_r (\!|\Gamma'|\!) \in \Gamma$ which, according to P1, is a contradiction and therefore $\max L(\Gamma, \Gamma') \leq \min M(\Gamma, \Gamma')$. □

Proof (Proof of Lemma 26). Let $\rho \in \mathcal{L}$ and $\varphi \in \mathcal{L}[\rho]$ be formulae. The proof is by induction on φ. The boolean cases are trivial. For the case $\varphi = L_r \psi$, we proceed as follows.

($\Longrightarrow$) Suppose $L_r \psi \in \Gamma$ implying by Q2 that $L_0 \psi \in \Gamma$. We first show that $[\![\psi]\!] \neq \emptyset$, which can be established by contradiction using P1, P2, Q1, and Q9.

Suppose towards a contradiction that $\theta\,(\Gamma)\,([\![\psi]\!]) = \emptyset$ implying that $\neg L_0 (\!|\Gamma'|\!) \in \Gamma$ for all $\Gamma' \in [\![\psi]\!]$. Because $[\![\psi]\!]$ is finite, it can be enumerated as $[\![\psi]\!] = \{\Gamma_0, \Gamma_1, \ldots, \Gamma_n\}$. By P1 we must have either $L_r (\!|\Gamma_0|\!) \vee L_r (\!|\Gamma_1|\!) \vee \cdots \vee L_r (\!|\Gamma_n|\!) \in \Gamma$ or $\neg (L_r (\!|\Gamma_0|\!) \vee L_r (\!|\Gamma_1|\!) \vee \cdots \vee L_r (\!|\Gamma_n|\!)) \in \Gamma$. However, in the first case we get $L_0 (\!|\Gamma'|\!) \in \Gamma$, which is a contradiction, and in the second case we get $\neg L_r \psi \in \Gamma$, which is also a contradiction.

Suppose towards a contradiction that $\theta^-\,(\Gamma)\,([\![\psi]\!]) < r$ implying the existence of $\Gamma' \in [\![\psi]\!]$ such that $\Gamma \xrightarrow{x}_\rho \Gamma'$ where $x < r$, implying further by Lem. 38 that $\max L(\Gamma, \Gamma') < r$. This implies the existence of some $q \in R_\rho$, $q < r$ such that $L_q (\!|\Gamma'|\!) \in \Gamma$ which, by Q2, implies that $L_0 (\!|\Gamma'|\!) \in \Gamma$. By the inductive hypothesis, we must have $\psi \in \Gamma'$ and therefore we get $(\!|\Gamma'|\!) \to \psi \in \Gamma'$ by P2. Q8 then yields that $L_r (\!|\Gamma'|\!) \in \Gamma$ which, since $\max L(\Gamma, \Gamma') < r$, is a contradiction and therefore $\theta^-\,(\Gamma)\,([\![\psi]\!]) \geq r$ implying $\mathcal{M}_\rho, \Gamma \models L_r \psi$.

($\Longleftarrow$) Suppose $\mathcal{M}_\rho, \Gamma \models L_r \psi$ implying $\theta^-\,(\Gamma)\,([\![\psi]\!]) \geq r$. This means that $\theta\,(\Gamma)\,([\![\psi]\!]) \neq \emptyset$ and hence $[\![\psi]\!] \neq \emptyset$. We can partition $[\![\psi]\!]$ in the following way. Let $E = \{\Gamma' \in [\![\psi]\!] \mid L(\Gamma, \Gamma') = \emptyset\}$ and $N = \{\Gamma' \in [\![\psi]\!] \mid L(\Gamma, \Gamma') \neq \emptyset\}$, then we have $E \cup N = [\![\psi]\!]$, $E \cap N = \emptyset$, $\neg L_0 (\!|\Gamma'|\!) \in \Gamma$ for all $\Gamma' \in E$, and $L_r (\!|\Gamma'|\!) \in \Gamma$ for all $\Gamma' \in N$. Lemma 35 then gives

$$\bigwedge_{\Gamma' \in E} \neg L_0 (\!|\Gamma'|\!) \wedge \bigwedge_{\Gamma' \in N} L_r (\!|\Gamma'|\!) \in \Gamma \ .$$

By Q3, this implies that

$$\bigwedge_{\Gamma' \in E} \neg L_0 (\!|\Gamma'|\!) \wedge L_r \bigvee_{\Gamma' \in N} (\!|\Gamma'|\!) \in \Gamma \ ,$$

so Q5 gives

$$L_r \bigvee_{\Gamma' \in [\![\psi]\!]} (\!|\Gamma'|\!) \in \Gamma \ .$$

By the induction hypothesis, Lem. 37, and Lem. 36, this implies that $L_r\psi \in \Gamma$.

The M_r case is similar, using Q7 instead of Q2 to establish that $L_0\psi \in \Gamma$. □

Proof (Proof of Theorem 27). ($1 \implies 2$) Suppose there exists a maximal set $\Gamma \in 2^{\mathcal{L}[\rho]}$ such that $\rho \in \Gamma$. We can then construct the WTS $\mathcal{M}_\rho$ given by Def. 25. Then by Lem. 26 we have $\mathcal{M}_\rho, \Gamma \models \rho$.

($2 \implies 1$) Suppose there exists some model $\mathcal{M} = (S, \rightarrow, \ell)$ such that $\mathcal{M}, s \models \rho$ for some $s \in S$. Let $\Gamma_m = \{\varphi \in \mathcal{L}[\rho] \mid \mathcal{M}, m \models \varphi\}$. Clearly $\rho \in \Gamma_m$. It remains to be shown that Γ_m is maximal.

P1: $\varphi \in \Gamma_m$ iff $\mathcal{M}, m \models \varphi$ iff $\mathcal{M}, m \not\models \neg\varphi$ iff $\neg\varphi \notin \Gamma_m$.
P2: If $\varphi \wedge \psi \in \Gamma_m$, then $\mathcal{M}, m \models \varphi \wedge \psi$, meaning that $\mathcal{M}, m \models \varphi$ and $\mathcal{M}, m \models \psi$. Hence $\varphi \in \Gamma_m$ and $\psi \in \Gamma_m$.
P3: If $\varphi \vee \psi \in \Gamma_m$, then $\mathcal{M}, m \models \varphi \vee \psi$, meaning that $\mathcal{M}, m \models \varphi$ or $\mathcal{M}, m \models \psi$. Hence $\varphi \in \Gamma_m$ or $\psi \in \Gamma_m$.

We have thus proven that Γ_m is propositionally maximal. The fact that Γ_m is quantitatively maximal can be proven in exactly the same way as in the proof of Thm. 15. Hence Γ_m is maximal. □

References

1. Larsen, K.G., Mardare, R., Xue, B.: Concurrent weighted logic. J. Log. Algebr. Meth. Program. 84 (2015)